Modelling the Way Mathematics Is Actually Done

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The problem:

- Whereas formal mathematical theories are well studied, computers cannot yet adequately represent and reason about mathematical dialogues and other informal texts.
- Machine learning is likely to be useful for building mathematical AI.
- But for that we need representations of mathematics in which meaningful patterns can be found.
Background

**Formal register:**

“Every integer equals the sum of four squares.”

\[ \forall n \in \mathbb{N} \exists m_1, m_2, m_3, m_4 \in \mathbb{N} \ n = \sum_{i=1}^{4} m_i^2 \]

- Nothing essential is lost in translating between the verbal and symbolic statements ("no reference is made … to meaning").
- *Trees* provide the look and feel of the formal register.

**Expository register:**

“Next, we will prove the four-square theorem using an algebraic identity similar to the one we just used to prove the two squares theorem.”
Cities are not trees

– Christopher Alexander
Cities can be *imagined* without overlapping systems...
Framing the Current Effort

The blocks world, board games, and story comprehension require increasingly sophisticated patterns of **inference**, **thinking**, and **reasoning**.

<table>
<thead>
<tr>
<th>Level</th>
<th>Blocks World</th>
<th>Board Games</th>
<th>Story Comprehension</th>
</tr>
</thead>
<tbody>
<tr>
<td>elements</td>
<td>blocks on a table</td>
<td>game pieces on board</td>
<td>episodes from everyday life</td>
</tr>
<tr>
<td>inference</td>
<td>follow instructions</td>
<td>rules &amp; strategy</td>
<td>analogy</td>
</tr>
<tr>
<td>thinking</td>
<td>consistency</td>
<td>prediction of winning</td>
<td>costs and benefits</td>
</tr>
<tr>
<td>reasoning</td>
<td>(trivial)</td>
<td>multiple strategies</td>
<td>ethical dilemmas</td>
</tr>
</tbody>
</table>

Understood as a computational challenge, mainstream mathematics lies somewhere in between board games and story comprehension.
Survey of Related Work

Annotative programming: Flare, ZigZag, AtomSpace

Models of Mathematical Reasoning:

1. Inference Anchoring Theory + Content 😊
2. Conceptual Dependence 😊
3. Structured Proofs 😐
4. Lakatos Games 😿
Inference Anchoring Theory + Content

This is what we use to model what people say when they talk about mathematics.
### IATC: Partial specification

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assert (s [, a])</td>
<td>Assert belief that statement (s) is true, optionally because of (a).</td>
</tr>
<tr>
<td>Agree (s [, a])</td>
<td>Agree with a previous statement (s), optionally because of (a).</td>
</tr>
<tr>
<td>Challenge (s [, a])</td>
<td>Assert belief that statement (s) is false, optionally because of (a).</td>
</tr>
<tr>
<td>Retract (s [, a])</td>
<td>Retract a previous statement (s), optionally because of (a).</td>
</tr>
<tr>
<td>Define (o, p)</td>
<td>Define object (o) via property (p).</td>
</tr>
<tr>
<td>Suggest (s)</td>
<td>Suggest a strategy (s).</td>
</tr>
<tr>
<td>Judge (s)</td>
<td>Apply a heuristic value judgement (s) to some statement.</td>
</tr>
<tr>
<td>Query (s)</td>
<td>Ask for the truth value of statement (s).</td>
</tr>
<tr>
<td>QueryE ({p_i(X)} . i)</td>
<td>Ask for the class of objects (X) for which all of the properties (p_i) hold.</td>
</tr>
<tr>
<td>has_property (o, p)</td>
<td>Object (o) has property (p).</td>
</tr>
<tr>
<td>strategy (m, s)</td>
<td>Method (m) may be used to prove (s).</td>
</tr>
<tr>
<td>beautiful (s)</td>
<td>Statement (s) is beautiful.</td>
</tr>
</tbody>
</table>
Conceptual Dependence

CD was used by Schank, Lytinen, and others to represent knowledge about actions, and to *reason about stories*.

“Willa was hungry. She picked up the Michelin guide.” (Why?)

CD data structures are generalised in Arxana.

Using something like CD, a system might reason about *why* people say what they do when they talk about mathematics.
Structured Proofs

This semi-formal style of writing down proofs, due to Lamport, is not all that well suited to describing *informal* reasoning.

**Proof Sketch:** What is the 500th digit of \((\sqrt{2} + \sqrt{3})^{2012}\)? Even this, eventually, a computer will be able to solve. The fact that this has been set as a problem is a huge clue. Can we do this for \(x + y\)? For \(e\)? Small rationals? And how about small perturbations of these? Maybe it is close to a rational?

\(1\) 1. For \(n\) large enough and \(m\) small enough in comparison, the \(m\)th digit of a sufficiently small rational \(r\) to the \(n\)th power is equal to 0.

**Proof:**

**Case:** \(r < 1/10\)

\(2\) 1. \((1/10)^n = .1^n\) has 0 in \(n - 1\) places in its decimal expansion, so we need \(m < n\).

\(2\) 2. \(r < 1/10\) implies \(r^n\) has zeros in at least \(n - 1\) places in its decimal expansion. □

\(1\) 2. Can we compute the \(m\)th digit of \((\sqrt{2} + \sqrt{3})^n\)?

**Proof:**

**Case:** \(n = 2\)

\(2\) 1. \((\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{2}\sqrt{3} + 3 \ast \)

\(2.1\)
This is a *formalised* description of informal reasoning, with a constrained structure. It’s plausible – but not sufficient.
The Search for the ‘Quantum of Progress’

Ganesalingam and Gowers’s ROBOTONE:

\[ \text{can ... be regarded as repeatedly applying a single tactic, which is itself constructed by taking a list of subsidiary tactics and applying the first that can be applied.} \]

Contrast this with Sussman’s classic program, HACKER.

\[ \text{In fact, Hacker is not as good at solving blocks world problems as would be a much simpler program that just goes about it directly with some good heuristics and a minimum of exploration. Hacker’s justification is as an epistemological model, not as a real problem solver.} \]

We saw part of this before.
IATC Example

NB. Pointing to edges
IATC Example

NB. Pointing to a subgraph

$\sqrt{2} + \sqrt{3})^{2012} + (\sqrt{3} - \sqrt{2})^{2012}$

And $(\sqrt{3} - \sqrt{2})^{2012}$ is a very small number. Maybe the final answer is "9"?

We need to check whether it's small enough.

$$(\sqrt{3} - \sqrt{2})^{2012} < (1/2)^{2012} = ((1/2)^{4})^{503} = (1/16)^{503} < .1^{503},$$

so we're in luck.
NB. At least one relevant edge is not drawn.
Towards Functional Models of Math. Reasoning

(Assert "contains as summand" 
  "(sqrt(2)+sqrt(3))^2012 
  +(sqrt(3)-sqrt(2))^2012"
  "(sqrt(3)-sqrt(2))^2012")
(Assert (has_property "(sqrt(3)-sqrt(2))^2012" 
  "is small"))
(Assert (implements #SUBGRAPH 
  "the trick might be: it 
  is close to something 
  we can compute"))
(Suggest (strategy "numbers that are very close 
  to integers have "9" 
  in many places of their 
  decimal expansion"))

S-expressions like those at left can be fed to Arxana, building up a graph representation.

But what about the reasoning that takes us from step to step?

Cf. Oxford Calculators, 14th C., *kinematics vs dynamics*
Arxana: *poly*graphs and nested semantic networks

LISP’s basic data structure: *cons cell* \((a . b)\), *car*, *cdr*
Arxana’s basic data structure: *nema* \((a c b)\), *src*, *txt*, and *snk*.

A repository of nemas is a *plexus*. \((0 a 0)\) is used to represent \(a\).
“Reified triples” by another name, but now with LISP inside!

Mom resents the fact that John disapproves of Jane and Jim’s marriage.

(example c/o Pierre De Lacaze)
Key ideas in the proof

“Why is 9 seen as a likely answer once we know that \((\sqrt{3} - \sqrt{2})^{2012}\) is small?”

\[
(\sqrt{3} + \sqrt{2})^{2012} + (\sqrt{3} - \sqrt{2})^{2012}
\]

equals

\(n\)

isa

integer

\((\sqrt{3} + \sqrt{2})^{2012}\)

has.property

9s.decimal

implies

suggests

strategy

implies

close.to

integer
One small reasoning step

In the paper, Listing 2 gives s-expressions detailing one step in the proof: the validation of a certain implements link.

The pictures on this slide and the ones following show what’s going on in Listing 2.
Along with the knowledge expressed in the proof itself, we assume that a suitable knowledge base is available to the system.\(^4\)

\(^4\)HDM stands for Hyperreal Dictionary of Mathematics project; ask me later.
One small reasoning step

One of the more exciting features of reasoning with Arxana is that we can encode inference rules in a *graph grammar*.

Here are the inference rules used to obtain the certificate:
One small reasoning step

Lastly, here is the certificate itself as a tree, i.e., a lambda expression, sitting inside of the `implements` node.

*Caveat*: this derivation was constructed by hand – the higher order reasoning required to select the premises, knowledge base elements, and inference rules, and to hook them all together in the correct way is not yet programmed!
Conclusions and Future Work

We have focused on a computational theory of the expository register. We draw upon contemporary argumentation theory and classic story understanding approaches in AI.

Future work may integrate themes from formal proof, embodiment and cognitive science, linguistics and NLP, as well as machine learning. Extensions to the system itself are planned to facilitate stepping through the challenge described earlier.

“It seems probable that once the machine thinking method had started, it would not take long to outstrip our feeble powers. There would be no question of the machines dying, and they would be able to converse with each other to sharpen their wits.”