Modelling the Way Mathematics Is Actually Done

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9 September, 2017
Workshop on Functional Art, Music, Modeling and Design
FARM'17 @ ICFP, Oxford, UK

The problem:

- Whereas formal mathematical theories are well studied, computers cannot yet adequately represent and reason about mathematical dialogues and other informal texts.
- Machine learning is likely to be useful for building mathematical AI.
- But for that we need representations of mathematics in which meaningful patterns can be found.

Background

Formal register:

"Every integer equals the sum of four squares."

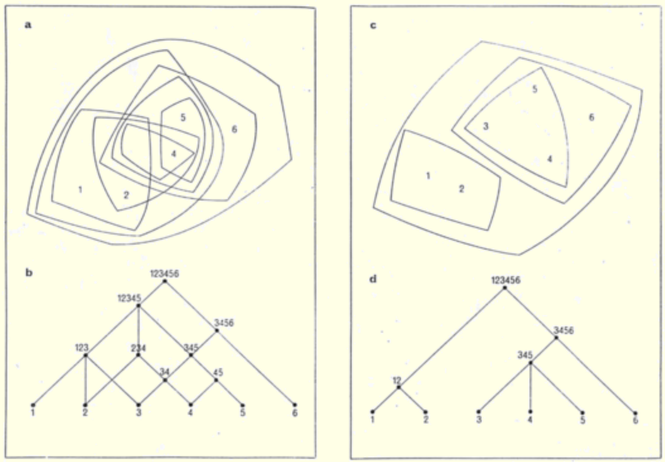
$$\equiv \quad (\forall n \in \mathbb{N})(\exists m_1, m_2, m_3, m_4 \in \mathbb{N})\, n = \sum_{i=1}^{4} m_i^2$$

- Nothing essential is lost in translating between the verbal and symbolic statements ("no reference is made … to meaning").
- *Trees* provide the look and feel of the formal register.

Expository register:

"Next, we will prove the four-square theorem using an algebraic identity similar to the one we just used to prove the two squares theorem."

Cities are not trees



– Christopher Alexander

Cities can be *imagined* without overlapping systems…

Framing the Current Effort

The blocks world, board games, and story comprehension require increasingly sophisticated patterns of *inference*, *thinking*, and *reasoning*.

Level	**Blocks World**	**Board Games**	**Story Comprehension**
elements	blocks on a table	game pieces on board	episodes from everyday life
inference	follow instructions	rules & strategy	analogy
thinking	consistency	prediction of winning	costs and benefits
reasoning	(trivial)	multiple strategies	ethical dilemmas

↑

Understood as a computational challenge, mainstream mathematics lies somewhere in between board games and story comprehension.

Survey of Related Work

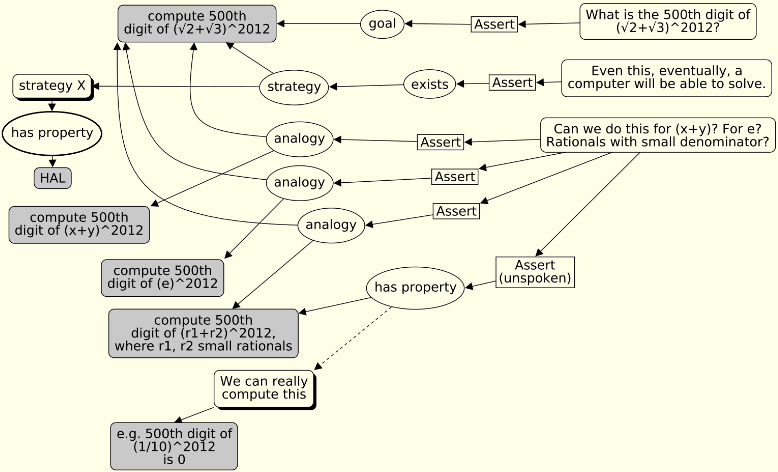
Annotative programming: *Flare*, *ZigZag*, *AtomSpace*

Models of Mathematical Reasoning:

1. Inference Anchoring Theory + Content ☺
2. Conceptual Dependence ☺
3. Structured Proofs 😐
4. Lakatos Games 😼

Inference Anchoring Theory + Content

This is what we use to model what people say when they talk about mathematics.



IATC: Partial specification

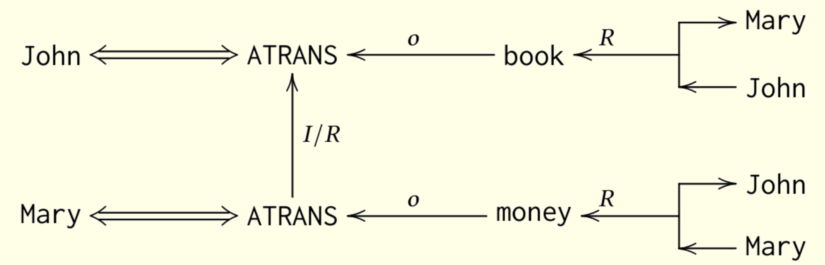
Assert (s [, a])	Assert belief that statement s is true, optionally because of a.
Agree (s [, a])	Agree with a previous statement s, optionally because of a.
Challenge (s [, a])	Assert belief that statement s is false, optionally because of a.
Retract (s [, a])	Retract a previous statement s, optionally because of a.
Define (o, p)	Define object o via property p.
Suggest (s)	Suggest a strategy s.
Judge (s)	Apply a heuristic value judgement s to some statement.
Query (s)	Ask for the truth value of statement s.
QueryE ($\{p_i(X)\} \,.\, i$)	Ask for the class of objects X for which all of the properties p_i hold.
has_property (o, p)	Object o has property p.
strategy (m, s)	Method m may be used to prove s.
beautiful (s)	Statement s is beautiful.

Conceptual Dependence

CD was used by Schank, Lytinen, and others to represent knowledge about actions, and to *reason about stories*.

"Willa was hungry. She picked up the Michelin guide." (Why?)

CD data structures are generalised in Arxana.



Using something like CD, a system might reason about why people say what they do when they talk about mathematics.

Structured Proofs

This semi-formal style of writing down proofs, due to Lamport, is not all that well suited to describing *informal* reasoning.

PROOF SKETCH: What is the 500th digit of $(\sqrt{2}+\sqrt{3})^{2012}$? Even this, eventually, a computer will be able to solve. The fact that this has been set as a problem is a huge clue. Can we do this for $x+y$? For e? Small rationals? And how about small perturbations of these? Maybe it is close to a rational?

⟨1⟩1. For n large enough and m small enough in comparison, the mth digit of a sufficiently small rational r to the nth power is equal to 0. **1**

PROOF:

CASE: $r < 1/10$

⟨2⟩1. $(1/10)^n = .1^n$ has 0 in $n-1$ places in its decimal expansion, so we need $m < n$. **1.1**

⟨2⟩2. $r < 1/10$ implies r^n has zeros in at least $n-1$ places in its decimal expansion. □ **1.2**

⟨1⟩2. Can we compute the mth digit of $(\sqrt{2}+\sqrt{3})^n$? **2**

PROOF:

CASE: $n = 2$

⟨2⟩1. $(\sqrt{2}+\sqrt{3})^2 = 2+2\sqrt{2}\sqrt{3}+3$ * **2.1**

…

Lakatos Games

This is a *formalised* description of informal reasoning, with a constrained structure. It's plausible – but not sufficient.

S1	P. Conjecture($\boldsymbol{c}$)
S2.2	P. Lemma($\boldsymbol{l}$)
S3.1	P. Lemma
S3.2	P. ProofDone
S2.1	P. ProofDone() (option only when $\mathcal{L} \neq \emptyset$)
S4.1	O. GlobalCounter($\boldsymbol{m}, c$)
S5.1	P. PiecemealExclusion($\boldsymbol{b}$)
S8.1	P. ProofDone
S8.2	P. Lemma
S5.2	P. StrategicWithdrawal
S5.3	P. MonsterBar($m, c, \boldsymbol{r}$)
S10	P. PDefinition($m, r, \boldsymbol{d}$)
S12.1	O. MonsterAccept(m, r)
S13.1	O. GlobalCounter
S13.2	O. LocalCounter
S13.3	O. HybridCounter
S13.4	P. StrategicWithdrawal
S13.5	O. Accept
S12.1	O. MonsterReject($m, r, d, \boldsymbol{s}, c$)
S14	O. ODefinition($m, r, d, s, \boldsymbol{e}$)
S15.1	O. Prefer(m, r, d, e)
S16.3	O. GlobalCounter
S16.3	O. LocalCounter
S16.3	O. HybridCounter
S16.3	P. StrategicWithdrawal
S16.3	O. Accept
S15.2	P. Prefer(m, r, e, d)
S16.2	P. PiecemealExclusion
S16.2	P. StrategicWithdrawal
S16.2	P. MonsterAdjust*
S16.2	P. GlobalLemmaInc
S16.2	P. Surrender
S5.4	P. MonsterAdjust($m, \boldsymbol{r}$)
S11	P. PDefinition($m, r, \boldsymbol{d}$)
S12.1	O. MonsterAccept
S12.2	O. MonsterReject($m, r, d, \boldsymbol{s}, c$)
S14	O. ODefinition($m, r, d, s, \boldsymbol{e}$)
S15.1	O. Prefer(m, r, d, e)
S16.3	O. GlobalCounter
S16.3	O. LocalCounter
S16.3	O. HybridCounter
S16.3	P. StrategicWithdrawal
S16.3	O. Accept
S15.2	P. Prefer(m, r, e, d)
S16.1	P. PiecemealExclusion
S16.1	P. StrategicWithdrawal
S16.1	P. MonsterBar*
S16.1	P. GlobalLemmaInc
S16.1	P. Surrender
S5.5	P. GlobalLemmaInc($m, \boldsymbol{k}$)
S17	P. HybridLemmaInc(m, k)
S18	P. Conjecture
S5.6	P. Surrender
S4.2	O. LocalCounter($\boldsymbol{m}, l$)
S6.1	P. LocalLemmaInc($m, l, \boldsymbol{k}$)
S19	P. ProofDone
S6.2	P. Surrender
S4.3	O. HybridCounter($\boldsymbol{m}, l, c$)
S7.1	P. HybridLemmaInc(m, l)
S18	P. Conjecture
S7.2	P. Surrender
S4.4	P. StrategicWithdrawal($\boldsymbol{s}, c$)
S9	P. Conjecture
S4.5	O. Accept

The Search for the 'Quantum of Progress'

Ganesalingam and Gowers's ROBOTONE:

> *can ... be regarded as repeatedly applying a single tactic, which is itself constructed by taking a list of subsidiary tactics and applying the first that can be applied.*[1]

Contrast this with Sussman's classic program, HACKER.[2]

> *In fact, Hacker is not as good at solving blocks world problems as would be a much simpler program that just goes about it directly with some good heuristics and a minimum of exploration. Hacker's justification is as an epistemological model, not as a real problem solver.*[3]

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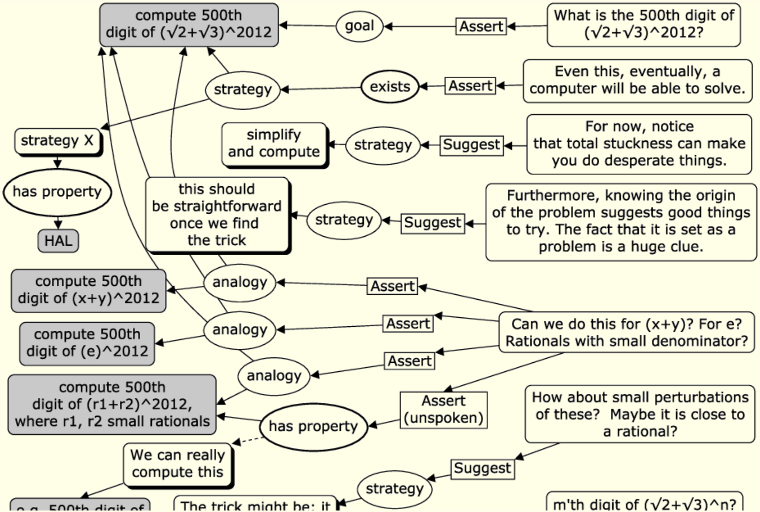
[1] M. Ganesalingam and W. T. Gowers. A Fully Automatic Theorem Prover with Human-Style Output. *Journal of Automated Reasoning*, pages 1–39, 2016.

[2] Gerald J. Sussman. A Computational Model of Skill Acquisition, PhD thesis, MIT, 1973.

[3] M. Levin. On Bateson's Logical Levels of Learning Theory. Tech. Rep. TM-57, MIT/LCS, 1975.

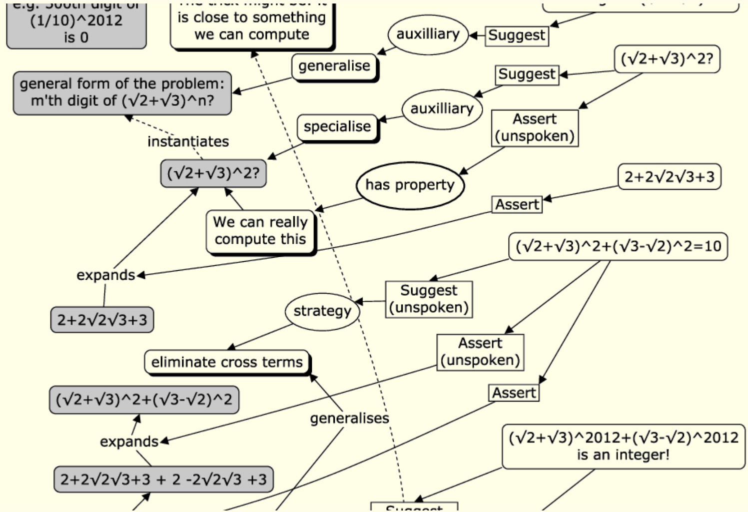
IATC Example

We saw part of this before.



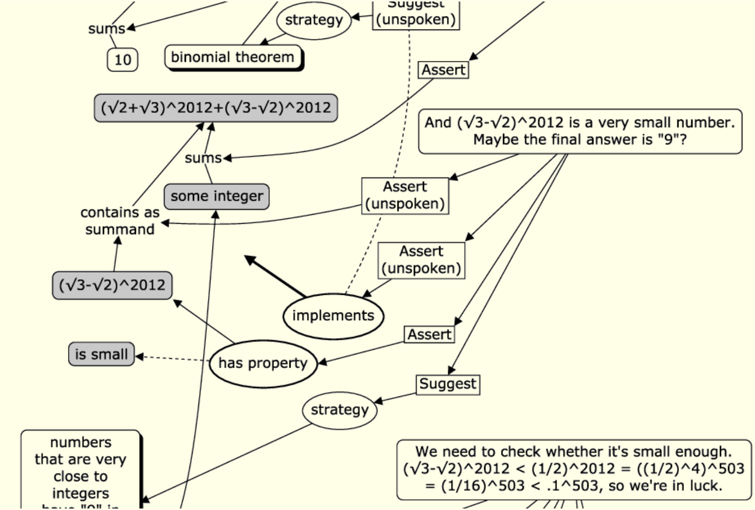
IATC Example

NB. Pointing to edges



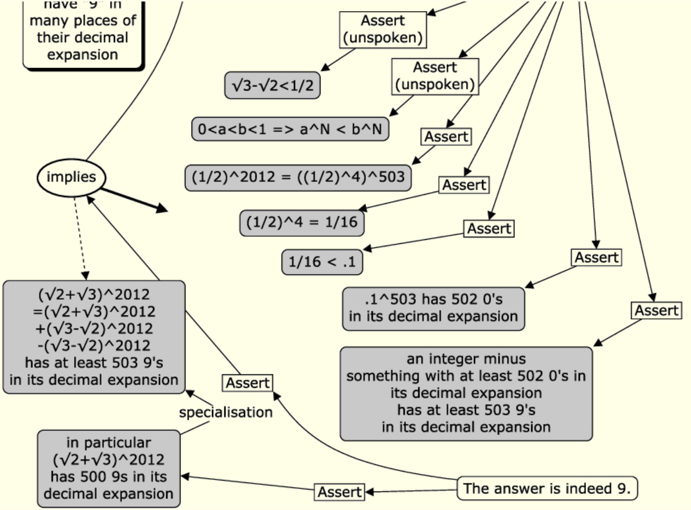
IATC Example

NB. Pointing to a subgraph



IATC Example

NB. At least one relevant edge is not drawn.



Towards Functional Models of Math. Reasoning

```
(Assert
"contains as summand"
"(sqrt(2)+sqrt(3))^2012
+(sqrt(3)−sqrt(2))^2012"
"(sqrt(3)−sqrt(2))^2012")
(Assert (has_property
"(sqrt(3)−sqrt(2))^2012"
"is small"))
(Assert (implements #SUBGRAPH
"the trick might be: it
is close to something
we can compute"))
(Suggest (strategy
"numbers that are very close
to integers have \"9\"
in many places of their
decimal expansion"))
```

S-expressions like those at left can be fed to Arxana, building up a graph representation.

But what about the reasoning that takes us from step to step?

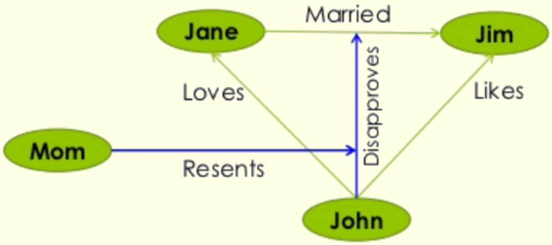
Cf. Oxford Calculators, 14th C., kinematics vs dynamics

Arxana: *poly*graphs and nested semantic networks

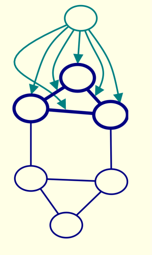
LISP's basic data structure: *cons cell* (a . b), car, cdr
Arxana's basic data structure: *nema* (a c b), src, txt, and snk.

A repository of nemas is a *plexus*. (0 a 0) is used to represent a.
"Reified triples" by another name, but now with LISP inside!

Mom resents the fact that John disapproves of Jane and Jim's marriage.



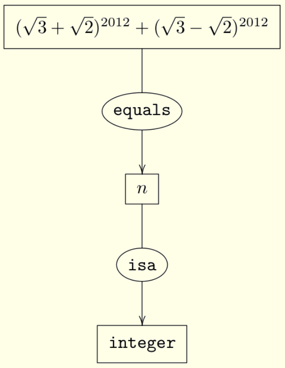
A "cone":

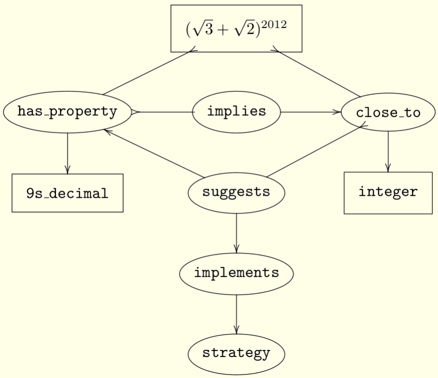


(example c/o Pierre De Lacaze)

Key ideas in the proof

"Why is 9 seen as a likely answer once we know that $(\sqrt{3}-\sqrt{2})^{2012}$ is small?"

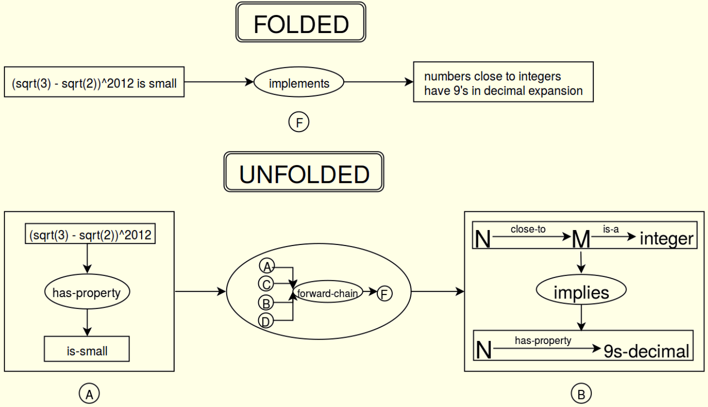




One small reasoning step

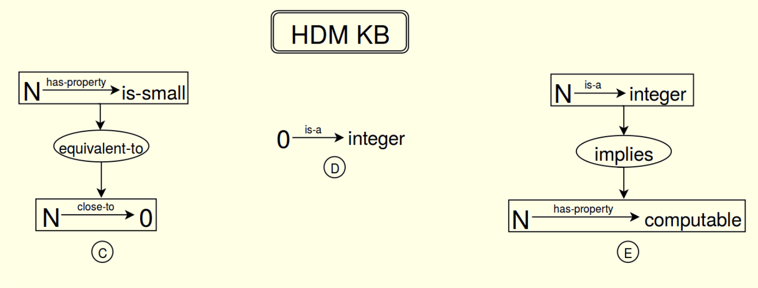
In the paper, Listing 2 gives s-expressions detailing one step in the proof: the validation of a certain implements link.

The pictures on this slide and the ones following show what's going on in Listing 2.



One small reasoning step

Along with the knowledge expressed in the proof itself, we assume that a suitable knowledge base is available to the system.[4]

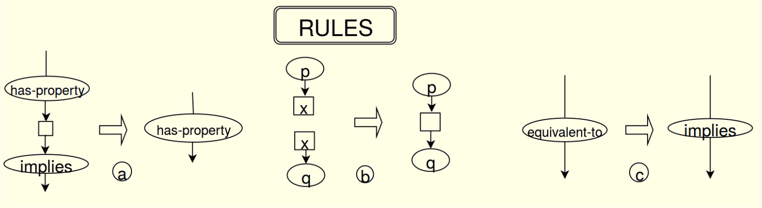


[4]HDM stands for *Hyperreal Dictionary of Mathematics* project; ask me later.

One small reasoning step

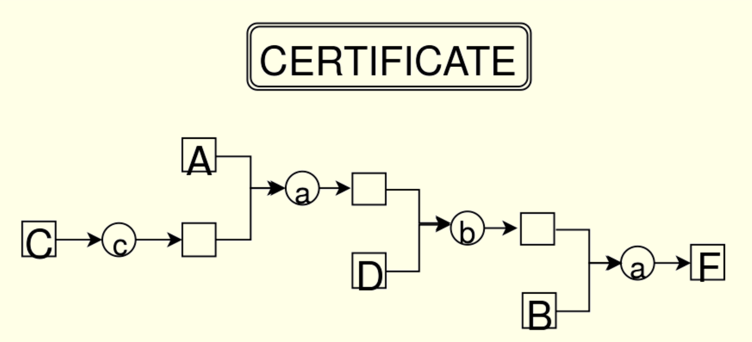
One of the more exciting features of reasoning with Arxana is that we can encode inference rules in a *graph grammar*.

Here are the inference rules used to obtain the certificate:



One small reasoning step

Lastly, here is the certificate itself as a tree, i.e., a lambda expression, sitting inside of the implements node.



Caveat: this derivation was constructed by hand – the higher order reasoning required to select the premises, knowledge base elements, and inference rules, and to hook them all together in the correct way is not yet programmed!

Conclusions and Future Work

We have focused on a computational theory of the expository register. We draw upon contemporary argumentation theory and classic story understanding approaches in AI.

Future work may integrate themes from formal proof, embodiment and cognitive science, linguistics and NLP, as well as machine learning. Extensions to the system itself are planned to facilitate stepping through the challenge described earlier.

"It seems probable that once the machine thinking method had started, it would not take long to outstrip our feeble powers. There would be no question of the machines dying, and they would be able to converse with each other to sharpen their wits."